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ON THE EXPONENTIAL FUNCTION AND PÓLYA'S PROOF [1]

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Let $a_1, \cdots, a_n$ be positive numbers. Denote by $A = \frac{1}{n} \sum_{i=1}^{n} a_i$ and $G = \left[\prod_{i=1}^{n} a_i \right]^{1/n}$. Pólya [1, pp. 103] has given an elementary proof that $A \geq G$ as follows

$$1 = e^0 = \exp\left[\sum_{i=1}^{n} \left(\frac{a_i}{A} - 1 \right) \right] \geq \prod_{i=1}^{n} \frac{a_i}{A} = \frac{G^n}{A^n}.$$

Wetzel [2] pointed out the two properties essential for the proof characterize the exponential function namely

$$f(x+y) \geq f(x)\, f(y) \tag{1}$$

$$f(x) \geq 1 + x. \tag{2}$$

Theorem (Wetzel). *Let* f *be defined on an interval containing the origin and such that* f *satisfies (1) and (2) on* I *then* $f(x) = e^x$.

If we consider functions of several variables then the analogues of (1) and (2) are

$$f(x+y) \geq f(x)\, f(y) \tag{3}$$

$$f(x) \geq \prod_{i=1}^{n} (1 + x_i) \tag{4}$$

(1) Received April, 1974.

where $x=(x_1,\cdots,x_n)$, $y=(y_1,\cdots,y_n)$, $x+y=(x_1+y_1,\cdots,x_n+y_n)$. Unfortunately these are not sufficient for Pólya's proof and we replace (3) by

$$f(x)=1 \quad \text{for all} \quad x=(x_1,\cdots,x_n) \quad \text{with} \quad \sum_{i=1}^{n} x_i=0. \tag{5}$$

From (5), with $x_i=\dfrac{a_i}{A}-1$ we have $1=f(x)$, then from (4) we have $1\geq\prod_{i=1}^{n}\left(1+\dfrac{a_i}{A}-1\right)=\dfrac{G^n}{A^n}$. It seems natural to ask whether (4) and (5) characterize the n-variable exponential function $\exp\left(\sum_{i=1}^{n}x_i\right)$. The answer is no as is shown by the following lemma.

LEMMA. *Let* $-1<x_i<1$ *for* $i=1,2,\cdots,n$ *and*

$$1-\left(\sum_{i=1}^{n-1}x_i\right)>0$$

then

$$f(x)\geq\prod_{i=1}^{n}(1+x_i) \tag{4}$$

$$f(x)=1 \quad \textit{for} \quad x_1+\cdots+x_n=0, \quad x=(x_1,\cdots,x_n) \tag{5}$$

where

$$f(x)=\frac{1+x_n}{1-\sum_{i=1}^{n-1}x_i}\neq\exp\left(\sum_{i=1}^{n}x_i\right).$$

PROOF. We proceed by induction. Consider $n=2$, since $-1<x_1<1$, $1>1-x_1^2>0$ and $1+x_2>0$ hence $\dfrac{1}{1-x_1}>1+x_1$

and $\dfrac{1+x_2}{1-x_1} \geq (1+x_1)(1+x_2)$. If $x_1 + x_2 = 0$ then $x_1 = -x_2$ and $1 - x_1 = 1 + x_2$ hence $\dfrac{1+x_2}{1+x_2} = 1$. Suppose now that $\dfrac{1+x_k}{1-\sum_{i=1}^{k-1} x_i} \geq \prod_{i=1}^{k}(1+x_i)$ or equivalently $\dfrac{1}{1-\sum_{i=1}^{k-1} x_i} \geq \prod_{i=1}^{k-1}(1+x_i)$ for all $k \leq n$, all $-1 < x_i < 1$, $i = 1, 2, \cdots, k$ and $1 - \left(\sum_{i=1}^{k-1} x_i\right) > 0$. Rewrite

$$\frac{1}{1-\sum_{i=1}^{k} x_i} = \left(\frac{1}{1-x_k}\right) \Big/ \left[1 - \sum_{i+1}^{k-1} \frac{x_i}{1-x_k}\right].$$

Since

$$1 - \sum_{i=1}^{k-1} \frac{x_i}{1-x_k} > 0, \quad \text{i. e.} \quad 1 - \sum_{i=1}^{k} x_i > 0$$

by the induction hypothesis

$$\frac{1}{1-\sum_{i=1}^{k} x_i} \geq \frac{1}{1-x_k} \cdot \prod_{i=1}^{k-1}\left[1 + \frac{x_i}{1-x_k}\right].$$

But $1 + \dfrac{x_i}{1-x_k} \geq 1 + x_i$ and $\dfrac{1}{1-x_k} \geq 1 + x_k$ hence

$$\frac{1}{1-\sum_{i=1}^{k} x_i} \geq \prod_{i=1}^{k}(1+x_i) \quad \text{or} \quad \frac{1+x_{k+1}}{1-\sum_{i=1}^{k} x_i} \geq \prod_{i=1}^{k+1}(1+x_i).$$

Clearly (5) is satisfied by f for $n \geq 2$.

We should note at this point that even (4) and (5) are stronger properties that are necessary for PÓLYA's proof. All that is really needed is the inequality

$$1 \geq \prod_{i=1}^{n}(1+x_i)$$

if $x_1 + \cdots + x_n = 0$. This of course is a simple consequence of

$$1 = e^0 = \exp \sum_{i=1}^{n} x_i \geq \prod_{i=1}^{n} \exp x_i$$

$$\geq \prod_{i=1}^{n} (1 + x_i).$$

However we can also give an elementary proof without using the exponential function.

LEMMA. *Let* $x_1, \cdots, x_n$ *be real numbers such that* $x_1 + \cdots + x_n = 0$ *then* $1 \geq \sum_{i=1}^{n} (1 + x_i)$.

PROOF. Since $x_1 + \cdots + x_n = 0$ if not all x_i's are zero, there is one such that $x_i < 0$. Without loss of generality assume $x_n < 0$. Consider $n = 2$ then $(1 + x_1)(1 + x_2) = (1 - x_1)(1 + x_1) = 1 - x_1^2 \leq 1$. Suppose then that for all $k \leq n - 1$ and $x_1 + \cdots + x_k = 0$, $1 \geq \prod_{i=1}^{k} (1 + x_i)$. Then $1 \geq \left[\prod_{i=1}^{n-2} (1 + x_i) \right] (1 + x_{n-1} + x_n)$ but $1 + x_{n-1} + x_n = (1 + x_n)\left(1 + \frac{x_{n-1}}{1 + x_n}\right)$. Since we also assume without loss of generality that $-1 < x_n$, we have $\frac{x_{n-1}}{1 + x_n} > x_{n-1}$ hence $1 + x_{n-1} + x_n \geq (1 + x_n)(1 + x_{n-1})$ and $1 \geq \prod_{i=1}^{n} (1 + x_i)$.

Hence we now could prove that $A \geq G$ without using the properties of the exponential function.

In conclusion we return to the characterization of the exponential function.

THEOREM. *Let* f *be defined on an open rectangle* I *in* R^n *such that the origin is in* I. *Suppose further that*

$$f(0) = 1 \qquad 0 = (0, \cdots, 0) \tag{6}$$

$$f(x + \hat{h}_i) \geq f(x)(1 + h_i) \tag{7}$$

where $\hat{h}_i = (0, \cdots, h_i, \cdots, 0)$ *for* $i = 1, \cdots, n$ *and* $x, \hat{h}_i,\ x + \hat{h}_i$ *in* I. *Then* $f(x) = \exp \sum_{i=1}^{n} x_i$.

PROOF. We consider first the case where $h_i > 0$. From (7) we have

$$f(x) \geq f(x + h_i)(1 - h_i)$$

or

$$f(x) \frac{1}{1 - h_i} \gtrsim \frac{f(x + h_i) - f(x)}{h_i}$$

and also $\dfrac{f(x + h_i) - f(x)}{h_i} \gtrsim f(x)$. If $h_i < 0$ both inequalities reverse. In either case we conclude that $\dfrac{f(x)}{x_i} = f(x)$, hence $f(x) = C \exp \sum_{i=1}^{n} x_i$. From (6) $C = 1$.

Note that (6) and (7) are «weaker» than (3) and (4).

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities*, 2nd Edition, Cambridge University Press, 1952.

[2] J. E. WETZEL, *On the functional equality* $f(x + y) \geqslant f(x) + (y)$, American Mathematical Monthly **74** (1967), pp. 1065.